MATH 2060 TUTO3
Thm 6-4-1 (Taylor's Thm)
Let · ne N
$\cdot f : [a,b] \longrightarrow \mathbb{R} s_{-t}. (a < b)$
· f' f ^m) are cts on [a,b] and
• $f^{(n+1)}$ exists on (a,b) .
If $x_0 \in (a,b)$, then $\forall x \in (a,b)$, $\exists c$ between x_0 and x s.t.
$\int (x) = \int (x^{\circ}) + \int (x^{\circ}) (x - x^{\circ}) + \cdots + \frac{1}{f(n)} (x^{\circ})} (x - x^{\circ}) + \frac{1}{f(n+1)} (x - x^{\circ})^{n+1}$
$P_n(x)$ $R_n(x)$
n-th Taylor's Polynomial of fat xo remainder
(Lagrange form)

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9. If $g(x) := \sin x$, show that the remainder term in Taylor's Theorem converges to zero as $n \to \infty$ for each fixed x_0 and x.

Ans: Note $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$ $\forall x \in \mathbb{R}$. So g is indefinitely differentiable and satisfies the conditions of Taylor's Thm for all $n \in \mathbb{N}$.

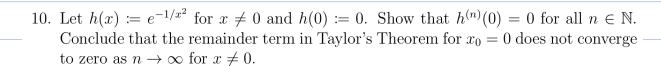
Fix x₀, x ∈ R.

The n-th remainder term in Taylor's Thm is $R_{n}(x) = \frac{g^{(n+1)}(C_{n})}{(n+1)!} (x-x_{0})^{n+1} \text{ for some } c_{n} \text{ between } x_{0}, x.$

Since $g^{(n+1)}(x) = \pm \sin x$ or $\pm \cos x$, we have $|R_n(x)| = \frac{|g^{n+1}(C_n)|}{(n+1)!} |x-x|^{n+1} \leq \frac{|x-x|^{n+1}}{(n+1)!} = \frac{1}{2} \alpha_n$

Want: $\lim_{n\to\infty} (a_n) = 0$. Use Ratio Test! Note $\lim_{n\to\infty} (\frac{a_{n+1}}{a_n}) = \lim_{n\to\infty} (\frac{|x-x_0|}{n+1})$ (assume $x_0 \neq x$) $= 0 < |x_0|$

By Ratio Test, $\lim_{n\to\infty} (a_n) = 0$ Therefore $\lim_{n\to\infty} (R_n(x)) = 0$ by Squeeze Thm.



So his smooth but not analytic (locally given by a power series)

Ans: Clearly
$$h(x)$$
 is infinitely diff for $x \neq 0$

Apply Leibniz rule to $h'(x) = \frac{2}{x^3} e^{-\sqrt{x^2}} = \frac{2}{x^7} h(x)$, we have

$$h''(x) = \frac{d^n}{dx^n} \left(\frac{2}{x^3} h(x)\right) = \frac{n}{k} \binom{n}{k} \binom{2}{x^3} h''(x)$$

$$= \sum_{k=0}^{n} {n \choose k} (2) (-3) (-4) \cdots (-(n-k+2)) \chi^{-(n-k+2)} \chi^{(k)}$$

$$= \sum_{k=0}^{n} {n \choose k} (-1)^{n-k} (n-k+2)! \frac{\lambda^{(k)}(x)}{x^{n-k+3}}$$

We prove by induction on n > 0 that

1)
$$\lim_{x \to 0} h^{(n)}(x)/x^m = 0$$
 $\forall m \in \mathbb{N}$

$$2) \qquad \sqrt{h^{(n+1)}(0)} = O.$$

When n = 0: 1) V m = N, by L'Hopital's rule,

$$\lim_{Y\to +\infty} \frac{y^m}{e^{\gamma}} = \lim_{Y\to +\infty} \frac{my^{m-1}}{e^{\gamma}} = \dots = \lim_{Y\to +\infty} \frac{m!}{e^{\gamma}} = 0$$

Hence
$$\lim_{x\to 0} \frac{h(x)}{x^m} = \lim_{x\to 0} \frac{(1/x^2)^m}{e^{1/x^2}} \cdot x^m = 0$$

$$\frac{2)}{h'(0)} = \lim_{x \to 0} \frac{h(x) - h(0)}{x - 0} = 0$$

Juppose 1), 2) are true for n

Now, by (*),

$$\lim_{k \to 0} \int_{k}^{h+k} (x) = \sum_{k=0}^{n} (n)(-1)^{k-k} (n-k+1)! \frac{1}{k!} \lim_{k \to 0} \int_{x}^{(k)} (x) \frac{1}{x^{n-k+1+m}} dx$$

Moveover, $\int_{x=0}^{(n+1)} \frac{1}{x^{n-k}} \frac{1}{x^{n-k}} \frac{1}{x^{n-k}} \frac{1}{x^{n-k+1+m}} dx$

This completes the induction.

Finally, $\lim_{x \to 0} \frac{1}{x^{n-k}} \frac{1}{x^{n-k}} \frac{1}{x^{n-k}} \frac{1}{x^{n-k}} \frac{1}{x^{n-k+1+m}} dx$

and so $\lim_{x \to 0} \frac{1}{x^{n-k}} \frac{1}{x^{n-k}} \frac{1}{x^{n-k}} \frac{1}{x^{n-k+1+m}} dx$

the inex is clearly fulse when X = 0.

11. If $x \in (0, 1]$ and $n \in \mathbb{N}$, show that

$$\left| \ln(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1} \frac{x^n}{n} \right) \right| < \frac{x^{n+1}}{n+1}.$$

Use this to approximate ln 1.5 with an error less than 0.01. Less than 0.001.

Ans: Let $f(t) = l_n(1+t)$.

Then f is infinitely diff. on (-1, a) and

$$f^{(n)}(t) = \frac{(-1)^{n-1}(n-1)!}{(1+t)^n} + -1, n \in \mathbb{N}$$

Fix x e (0,1) and n e IN.

By Taylor's Thm, $f(x) = P_n(x) + R_n(x)$,

 $P_{N}(x) = \sum_{k=0}^{N} \frac{f^{(k)}(0)}{k!} x^{k} = x - \frac{x^{k}}{x^{k}} + \frac{x^{k}}{x^{k}} - \cdots + (-1)^{k-1} \frac{x^{k}}{x^{k}}$

and for some $C_n \in (0, x)$,

 $R_{n}(x) = \frac{f(n+1)}{(n+1)!} x^{n+1} = \frac{1}{n+1} \cdot \frac{(-1)^{n}}{(1+C_{n})^{n+1}} x^{n+1}$

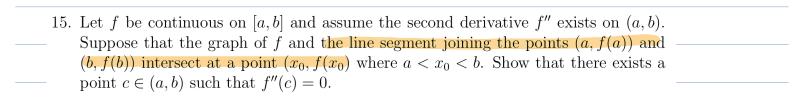
 N_{DW} , $|R_{\text{n}}(x)| < \frac{1}{n+1} \chi^{n+1}$

and so $\left| f(x) - (x - \frac{x^2}{3} + \frac{x^3}{3} - \dots + (-1)^{n-1} \frac{x^n}{n}) \right| = |R_n(x)| < \frac{1}{n+1} x^{n+1}$.

To approx $l_{1}(1.5)$, take X = 0.5. Then $|l_{1}(1.5) - P_{1}(0.5)| < \frac{1}{nt1} |0.5|^{nt1}$

When n=4, $\frac{1}{n+1}(0.5)^{n+1}=0.00625<0.01$.

So $I_{1}(1.5) \approx P_{4}(0.5) = \frac{77}{192}$ with error less than 0.01.



Ans: By assumption,

$$f(c_1) = \frac{f(b) - f(x_0)}{b - x_0} = \frac{f(x_0) - f(a)}{x_0 - a} = f'(c_1)$$

Apply MVT to f on [a, x.], = c, e(a, x.) s.t. $\frac{f(x_0) - f(\alpha)}{x_0 - \alpha} = f'(c_0)$

Apply MVT to f on [Xo, b], = Cre(xo, b) s.t. $\frac{f(b)-f(x_0)}{b-x_0}=f'(c_2)$

Note a < C1 < C2 < b,

f'' exists on $(a,b) \Rightarrow f'$ cts and diff. on $[C_1,C_2]$.

Apply MV7 again, $\exists c \in (c_1, c_2)$ s.t $f''(c) = \frac{f'(c_1) - f'(c_1)}{c_2 - c_1} = 0$

$$f''(c) = \frac{f'(c_1) - f'(c_1)}{c_2 - c_1} = 0$$

Example

Consider a function f whose second derivative f''(x) exists and is continuous on [0,1]. Assume that f(0) = f(1) = 0 and suppose that there exists A > 0 such that $|f''(x)| \le A$ for $x \in [0,1]$. Show that

$$\left| f'\left(\frac{1}{2}\right) \right| \le \frac{A}{4} \quad \text{and } |f'(x)| \le \frac{A}{2}$$

Ans:

Apply Taylor's Thm to f with center 1/2, we have $\forall x \in [0, 1]$, $\exists C_X$ between x, x s.t

$$f(x) = f(\frac{1}{2}) + f'(\frac{1}{2})(x - \frac{1}{2}) + \frac{f''(C_x)}{2!}(x - \frac{1}{2})^2$$

Put x = 1:

$$0 = f(1) = f(\frac{1}{2}) + f'(\frac{1}{2})(\frac{1}{2}) + f''(e_1)(\frac{1}{2^3})$$
Put x = 0:

$$| \uparrow (\overline{z}) | \leq \frac{1}{8} (| \uparrow (\overline{z}) | + | \uparrow (\overline{z}) |$$

$$\leq \frac{1}{8} (| A + A) = \frac{A}{4}$$

Fix QG (O11). Apply Taylor's Thm to f with center a, we have

$$\exists C_0 \in (D, a), \exists C_1 \in (a, 1) \quad s.t.$$

$$0 = f(1) = f(a) + f'(a)(1-a) + \frac{f''(C_1)}{2!}(1-a)^2 \quad (3)$$

$$0 = f(0) = f(a) + f'(a)(0-a) + \frac{f''(c_0)}{2!}(0-a)^2$$

$$3-4$$
: $f'(a) + \frac{1}{2} [f''(c)(|a|^2 + f''(c)a^2)]$

$$|f'(\alpha)| \leq \frac{A}{2} \left[(|-\alpha|^2 + \alpha^2) \right]$$

$$\leq \frac{A}{2} \left[(|-\alpha| + \alpha) \right] = \frac{A}{2}$$

If
$$\alpha = 0$$
, $3 \Rightarrow |f'(0)| = \frac{1}{2}|f''(c_1)| \leq \frac{1}{2}$
If $\alpha = 1$, $4 \Rightarrow |f'(1)| = \frac{1}{2}|f''(c_0)| \leq \frac{1}{2}$